

On a generalized Sierpiński fractal in \mathbb{RP}^n

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Abstract

We associate a fractal in \mathbb{RP}^n to each vector basis of \mathbb{R}^{n+1} and we study its measure and asymptotic properties. Then we discuss and study numerically in detail the cases $n = 1, 2, 3$, evaluating in particular their Hausdorff dimension.

1 Introduction

In this paper we study an algorithm that takes a basis of \mathbb{R}^{n+1} and builds, out of it, a fractal in \mathbb{RP}^n . We do this by using the following two basic facts: 1. In \mathbb{RP}^n , every $n + 1$ points $\{p_i\}$ which do not lie on the same affine hyperplane determine a partition of \mathbb{RP}^n in the 2^n projective n -simplices having the points p_i as vertices; 2. Given a vector basis $\mathcal{E} = \{e_i\}$ in \mathbb{R}^{n+1} , we can build $n + 1$ new bases \mathcal{E}_i by fixing the i -th vector e_i and summing it to the n remaining ones. Now, consider a vector basis \mathcal{E} of \mathbb{R}^{n+1} . Its vectors e_i projects into $n + 1$ points¹ $[e_i] \in \mathbb{RP}^n$ and therefore determine a partition of \mathbb{RP}^n in projective n -simplices as in point (1). We denote by $S(\mathcal{E})$ the one the point $[e_1 + e_2 + e_3]$. Next, consider the $n + 1$ projective n -simplices $S(\mathcal{E}_i)$ corresponding to the bases \mathcal{E}_i defined in (2). Their union $\cup_{i=1}^{n+1} S(\mathcal{E}_i)$ can be thought as the difference between $S(\mathcal{E})$ and the interior of the projective polytope $Z(\mathcal{E})$ (the *body* of \mathcal{E}) having the points $[e_i + e_j]$ as vertices. By repeating recursively this step on the $S(\mathcal{E}_i)$, we end up building a $(n + 1)$ -ary tree of bases $T(\mathcal{E}) = \{\mathcal{E}_I\}$, where $I = i_1 \dots i_k$ is a multiindex, and fractal $F(\mathcal{E})$ whose points are the ones left inside $S(\mathcal{E})$ after removing all the bodies $Z(\mathcal{E}_I)$.

Topologically, this fractal coincide with the multi-dimensional generalization of the Sierpinski triangle [Sie15], namely the fractal generated by removing from a n -simplex S the polytope Z having as vertices the middle points of the edges of S . Geometrically though they are different because the vertices $[e_i + e_j]$ of the body $Z(\mathcal{E})$ are closer to the vertices of $S(\mathcal{E})$ corresponding to the vectors of higher Euclidean norm, and even if we start with a basis where all vectors have the same norm they will not be anymore so after the first step.

¹Here and throughout the paper we denote by $[e] \in \mathbb{RP}^n$ the direction of the vector $e \in \mathbb{R}^{n+1}$.

We were motivated to study this fractal by the following two reasons. First, this fractal is invariant with respect to a set of $n + 1$ transformations (more specifically, projective diffeomorphisms) but they fail to be an Iterated Function System (IFS) because they are not contractions. In particular the machinery developed for IFSs cannot be applied to this case and no analytical bounds are known for the Hausdorff dimension of $F(\mathcal{E})$ (numerical estimates for the cases $n = 2, 3$ can be found in Section 4). Second, the construction leading from \mathcal{E} to $F(\mathcal{E})$ is the n -dimensional generalization of the structure discovered by the author and I.A. Dynnikov in a fractal, in \mathbb{RP}^2 , which describes the asymptotic behaviour of the plane sections of the triply-periodic cubic polyhedron $\mathcal{C} = \{4, 6|4\}$ [DD09]². Indeed one of the results of Novikov’s theory of plane sections of triply-periodic surfaces [Nov82, Dyn99] is the following. Consider a connected triply-periodic surface S which divides \mathbb{R}^3 in two components which are equal modulo translations. Then there exists no bundle of parallel planes whose intersections with S are all compact. The asymptotics of the open (i.e. non-compact) sections, as function of the direction of the bundle of planes, are described by a “labeled cut-out fractal” in the projective plane (see [De 03, De 06] for a few other concrete examples). Here by *labeled cut-out fractal* we mean a fractal F which is obtained by removing, from an initial region, a sequence of closed sets $\{Z_i\}$ whose interiors is pairwise disjoint and such that to each Z_i it is associated an element b_i of some set of “labels” B . In the particular case of Novikov’s theory above, the set of labels is the set of all indivisible triples of integers and the geometrical meaning of the fractal is the following. Let d be any vector in \mathbb{R}^3 whose direction $[d]$ belongs to some Z_i labeled by b_i . Then the open sections arising by cutting the surface S with planes perpendicular to d are strongly asymptotic to a straight line whose direction is given by the “vector product” $d \times b$. Going back to the surface \mathcal{C} , it turns out that its corresponding fractal $F_{\mathcal{C}}$ essentially coincides with the fractal $F(\mathcal{E}) \subset \mathbb{RP}^2$ associated to the basis $\mathcal{E} = \{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$ of \mathbb{R}^3 . We conclude this long digression by pointing out that the Novikov’s theory of plane sections of triply-periodic surfaces is the mathematical model for the phenomenon of the anisotropic behaviour of magnetoresistance in normal metals at low temperature and under a strong magnetic field (see [LP60] and [NM03] for more details about the physics and the dynamics of this phenomenon). In particular $F_{\mathcal{C}}$ encodes the information on the conduction of the electric current in a metal having \mathcal{C} as Fermi Surface.

The paper is organized as follows. In Section 2 we define the basic objects and prove a few elementary facts about them. In Section 3 we prove that all fractals $F(\mathcal{E})$ have zero volume with respect to some natural measure on the projective space and study their asymptotic properties, showing in particular that they are related with the n -bonacci sequences. Finally, in Section 4, we discuss in detail the cases $n = 1, 2, 3$ and present numerical results indicating that, unlike the Sierpiński case, the Hausdorff measure of $F(\mathcal{E})$ may be non-

²We discovered later that this fractal had been already considered in the past by G. Levitt [Lev93] while studying dynamical systems on the circle.

integer even for $n = 3$.

2 Structure of the fractal

Let $\mathcal{E} = \{e_1, \dots, e_{n+1}\}$ be a vector basis of \mathbb{R}^{n+1} and let us call *volume* of \mathcal{E} the Euclidean volume of the $(n+1)$ -simplex of \mathbb{R}^{n+1} naturally associated to it. Every of the $n+1$ sets $\mathcal{E}_j = \{e_1^j, \dots, e_{n+1}^j\}$, $j = 1, \dots, n+1$, defined by

$$\begin{cases} e_i^j = e_i + e_j, i \neq j \\ e_i^i = e_i \end{cases}$$

is also a vector basis of \mathbb{R}^{n+1} and has the same volume as \mathcal{E} . Repeating recursively this procedure, we get an infinite $(n+1)$ -ary ordered rooted tree $T(\mathcal{E}) = \{\mathcal{E}_{I_k}\}_{k \in \mathbb{N}}$ of bases of \mathbb{R}^{n+1} , all with the same volume, with \mathcal{E} as root. The multi-index $I_k = i_1 i_2 \dots i_{k-1} i_k$ describes the steps needed to build the basis from the root, namely $\mathcal{E}_{I_{k-1} i_k} = (\mathcal{E}_{I_{k-1}})_{i_k}$.

This tree structure corresponds to the limit process for building a fractal on \mathbb{RP}^n . Indeed let $\mathcal{E} = \{e_1, \dots, e_{n+1}\}$ and denote by $[e_i] \in \mathbb{RP}^n$ the direction of the vector e_i . To \mathcal{E} it is naturally associated a projective n -simplex $S(\mathcal{E})$ defined in the following way. The $n+1$ points $[e_i] \in \mathbb{RP}^n$ are the vertices of 2^n projective n -simplices whose interiors are pairwise disjoint and whose union gives the whole \mathbb{RP}^n ; the point $p = [e_1 + \dots + e_{n+1}]$ is not a boundary point for any of them and we denote by $S(\mathcal{E})$ the one which contains p . Now consider the bases \mathcal{E}_i at the first recursion level of $T(\mathcal{E})$. The vertices of the projective n -simplex $S(\mathcal{E}_i)$ are the points $\{[e_1 + e_i], \dots, [e_{i-1} + e_i], [e_i], [e_{i+1} + e_i], \dots, [e_{n+1} + e_i]\}$, i.e. $S(\mathcal{E}_i)$ is contained inside $S(\mathcal{E})$, shares with it the vertex $[e_i]$ and (part of) all the edges coming out from that point and has in common exactly one vertex with each other $S(\mathcal{E}_j)$, $j \neq i$.

Let now $F_1 = \cup_{i=1}^{n+1} S(\mathcal{E}_i)$. The difference between $S(\mathcal{E})$ and F_1 is the interior of the projective polytope $Z(\mathcal{E})$ having the $n(n+1)/2$ points $[e_i + e_j]$, $i \neq j$, as vertices. We call $Z(\mathcal{E})$ the *body* of \mathcal{E} . More generally, let $F_k = \cup_{|I|=k} S(\mathcal{E}_I)$ be the k -th level of recursion of the fractal, with $F_0 = S(\mathcal{E})$. The set F_k is obtained from F_{k-1} by erasing the interiors of the $(n+1)^{k-1}$ bodies $Z_J = Z(\mathcal{E}_J)$, $|J| = k-1$. The fractal $F(\mathcal{E})$ is then obtained as the limit $F(\mathcal{E}) = \cap_{k \in \mathbb{N}} F_k$.

Note that we can always find an affine n -plane (i.e. a canonical chart for the projective space) inside \mathbb{RP}^n which contains the entire $S(\mathcal{E})$ and, therefore, the whole $F(\mathcal{E})$. From now on then we will consider often $S(\mathcal{E})$ and all the $S(\mathcal{E}_I)$ as an n -simplex inside \mathbb{R}^n . This allows to provide another geometric characterization of the algorithm generating the fractal. Indeed the k -skeleton of $S(\mathcal{E})$ is the set of the convex hulls associated to the $\binom{n+1}{k}$ different subsets of k elements of \mathcal{E} , namely the convex hulls of the sets $\{[e_{i_1}], \dots, [e_{i_k}]\}$ where no two indices are equal.

Definition 1. The vector $b(\mathcal{E}) = \sum_{i=1}^{n+1} e_i \in \mathbb{R}^{n+1}$ is called the *barycenter* of the n -simplex $S(\mathcal{E})$, where $\mathcal{E} = \{e_i\}_{i=1, \dots, n+1}$. Analogously, the *barycenter* of its

k -face of vertices $\{[e_{i_1}], \dots, [e_{i_k}]\}$ is the vector $b_I(\mathcal{E}) = \sum_{j=1}^k e_{i_j} \in \mathbb{R}^{n+1}$, where $I = i_1 \dots i_k$. By abuse of notation we sometimes call barycenter its direction $[b] \in \mathbb{RP}^n$. It will be clear from the context which one we are referring to.

Lemma 1. *Let \mathcal{E} be a basis of \mathbb{R}^{n+1} and f_{I_k} the k -subsimplex of $S(\mathcal{E})$ corresponding to $\{e_{i_1}, \dots, e_{i_k}\} \subset \mathcal{E}$. Then the projection of the barycenter $[b_{I_k}]$ of f_{I_k} from the vertex $[e_{i_j}]$ on the $(k-1)$ -face of f_{I_k} opposite to it, namely the one corresponding to the $k-1$ vectors $\{e_{i_1}, \dots, e_{i_k}\} \setminus \{e_{i_j}\}$, coincides with the barycenter of that face.*

Proof. This relation is clearly recursive and therefore it is enough to prove the theorem in the case of the n -simplex $S(\mathcal{E})$ and any of its faces. Let us consider what happens for the vertex $[e_1]$: the face f_1 of $S(\mathcal{E})$ opposite to it corresponds, in \mathbb{R}^{n+1} , to the n -plane \hat{f}_1 spanned by the n vectors $\mathcal{E}^{(1)} = \{e_j\}_{j \neq 1}$ and the line l_1 joining $[e_1]$ to $[b]$ corresponds to the 2-plane \hat{l}_1 spanned by e_1 and b . Since $b = \sum_{k=2}^{n+1} e_k$, clearly the only linear combinations belonging to both \hat{f}_1 and \hat{l}_1 are the span of the vector $\sum_{k=2}^{n+1} e_k = b - e_1$. In other words, the intersection between f_1 and l_1 is $[e_2 + \dots + e_{n+1}]$, which is indeed the barycenter of $\mathcal{E}^{(1)}$ and similarly for the other vertices. \square

Proposition 1. *The vertices of the body $Z(\mathcal{E})$ corresponding to a basis \mathcal{E} can be obtained in the following way: project the barycenter of \mathcal{E} from its vertices to its faces and repeat recursively this procedure until the edges are reached. The $n(n+1)/2$ points obtained are the vertices of $Z(\mathcal{E})$.*

Proof. The recursive procedure makes sense because, thanks to the previous lemma, we know that the projection of the barycenter on a face via the vertex opposite to it coincides with the barycenter of the face. When we reach the edges, therefore, we are left with their barycenters, which are clearly the $n(n+1)/2$ points $[e_i + e_j]$, $i \neq j$. \square

Finally, we provide a third way to describe this fractal. Recall that the Sierpiński gasket and its natural multi-dimensional generalization can be seen the invariant set of a Iterated Functions Systems (IFS). Similarly, we prove below that the fractal $F(\mathcal{E})$ is the invariant set of $n+1$ projective diffeomorphisms $\{\psi_i\}$. They do not form however, strictly speaking, a IRS because they are not contractions; in particular the Jacobian of each of them is the identity in the omonymous vertex $[e_i]$ of $S(\mathcal{E})$.

Proposition 2. *The fractal $F(\mathcal{E})$ is invariant with respect to the $(n+1)$ projective automorphisms ψ_i of \mathbb{RP}^n induced by the linear transformations $\hat{\psi}_i$ defined by $\hat{\psi}_i(e_j) = e_i + e_j$, $j \neq i$, $\hat{\psi}_i(e_i) = e_i$.*

Proof. This is simply a consequence of the fact that the tree itself $T(\mathcal{E})$ is clearly invariant under the action of the $\hat{\psi}_i$, so that the ψ_i map the set of bodies $Z(\mathcal{E}_I)$ into itself and therefore leave the fractal invariant. \square

Remark 1. *Every body $Z_I(\mathcal{E})$, $|I| = k$, is the image of the root body $Z(\mathcal{E})$ via the map $\psi_I := \psi_{i_k} \circ \dots \circ \psi_{i_1}$.*

3 Measure and Asymptotics of the fractal

We start by proving that the volume of $F(\mathcal{E})$ is zero with respect to any measure μ induced on \mathbb{R}^n by the Lebesgue measure on any affine n -plane or, equivalently, with respect to the measure induced by the canonical one on the sphere.

We begin with a technical lemma:

Lemma 2. *The maximum $m_{k,n}$ of the functions*

$$f_{k,n}(v_1, \dots, v_n) = \frac{(1 + \sum_{i=1}^n v_i)^n}{(1 + \sum_{i=1}^n v_i + k(1 + \sum_{i=2}^n v_i))(1 + \sum_{i=1}^n v_i + (1+k)(1 + \sum_{i=2}^n v_i))^n}$$

where $k, n \in \mathbb{N}$, $n \geq 2$, on the n -simplex S with vertices in $p_0 = (0, \dots, 0)$, $p_1 = (1, 0, \dots, 0)$, \dots , $p_n = (0, \dots, 0, 1)$, is given by

$$m_{k,n} = f_{k,n}(1, 0, \dots, 0) = \frac{2^n}{(2+k)(3+k)^n}$$

except for the case $n = 2$, $k = 0$, where $m_{0,2} = f_{0,2}(0, 0, \dots, 0) = 1/4$

Proof. A direct computation shows that the derivative of $f_{k,n}$ with respect to any v_i , $i > 1$, is negative inside S and therefore the maximum is attained at the smallest values for those variables. Then we are left with the function

$$h_{k,n}(v_1) = \frac{(1 + v_1)^n}{(1 + k + v_1)(2 + k + v_1)^n}$$

whose derivative

$$h'_{k,n}(v_1) = \frac{(1 + v_1)^{n-1} \{k^2 n + (n - 2 - v_1)(1 + v_1) + k[n(2 + v_1) - 1 - v_1]\}}{(1 + k + v_1)^2 (2 + k + v_1)^{n+1}}$$

in the domain $v_1 \in (0, 1)$ is always positive for $k > 0$ while for $k = 0$ is always positive for $n > 2$ and always negative for $n = 2$.

Hence $f_{k,n}$ will reach its maximum in the origin when $k = 0$, $n = 2$ and in the point $(1, 0, \dots, 0)$ in all other cases. \square

The following proof is a generalization to any n of the proof provided in [DD09] for the case $n = 2$.

Theorem 1. *The fractal set $F(\mathcal{E})$ is a null set for μ .*

Proof. It is enough to prove the theorem for a particular choice of $\mathcal{E} = \{e_1 = (1, 0, \dots, 0, 1), \dots, e_n = (0, \dots, 0, 1, 1), e_{n+1} = (0, \dots, 0, 1)\}$. With this \mathcal{E} , the n -simplex $S(\mathcal{E})$ is contained in the affine plane $\pi = \{h_{n+1} \neq 0\}$ with respect

to the homogeneous coordinates $[h_1 : \dots : h_{n+1}]$. On π we use the canonical coordinates (v_1, \dots, v_n) defined by $v_i = h_i/h_{n+1}$ and the measure

$$d\mu' = \frac{dv_1 \cdots dv_n}{\left(1 + \sum_{i=1}^n |v_i|\right)^n}.$$

Each of the ψ_i maps the whole fractal $F = F(\mathcal{E}) \subset S(\mathcal{E})$ into disjoint sets $\psi_i(F) = F(\mathcal{E}_i) \subset S_i$ so that $\hat{\mu}(F) = \sum_{i=0}^{n+1} \hat{\mu}(\psi_i(F))$. With our choice of \mathcal{E} and $\hat{\mu}$ both $S(\mathcal{E})$ and the measure (at least close to $S(\mathcal{E})$) are invariant with respect to every permutation of the first n basis vectors and therefore $\hat{\mu}(\psi_i(F)) = \hat{\mu}(\psi_1(F))$, $\forall i = 1, \dots, n$, so that $\hat{\mu}(F) = n\hat{\mu}(\psi_1(F)) + \hat{\mu}(\psi_{n+1}(F))$. By repeating this procedure on $\psi_{n+1}(F)$ we find that

$$\hat{\mu}(\psi_{n+1}(F)) = n\hat{\mu}(\psi_{n+1} \circ \psi_1(F)) + \hat{\mu}(\psi_{n+1} \circ \psi_{n+1}(F))$$

so that $\hat{\mu}(F) = n\hat{\mu}(\psi_1(F)) + n\hat{\mu}(\psi_{n+1} \circ \psi_1(F)) + \hat{\mu}(\psi_{n+1} \circ \psi_{n+1}(F))$ and finally, by recursion,

$$\hat{\mu}(F) = n \sum_{k=0}^{\infty} \hat{\mu}(\psi_{n+1}^k \circ \psi_1(F))$$

since $\lim_{k \rightarrow \infty} \hat{\mu}(\psi_{n+1}^k(F)) = 0$.

We will now show that $\hat{\mu}(\psi_{n+1}^k \circ \psi_1(F)) \leq c_k^{(n)} \hat{\mu}(F)$ with $\sum_{k=0}^{\infty} c_k^{(n)} < 1/n$, which leads immediately $\hat{\mu}(F) = 0$. Note indeed that, with this particular choice of the basis, the action of the ψ_i on the corresponding homogeneous coordinates is given by

$$\begin{cases} \psi_1([h_1 : h_2 : \dots : h_n : h_{n+1}]) = [h_{n+1} : h_2 : \dots : h_n : 2h_{n+1} - h_1] \\ \vdots \\ \psi_n([h_1 : h_2 : \dots : h_n : h_{n+1}]) = [h_1 : h_2 : \dots : h_{n+1} : 2h_{n+1} - h_n] \\ \psi_{n+1}([h_1 : h_2 : \dots : h_n : h_{n+1}]) = [h_1 : h_2 : \dots : h_n : \sum_{i=1}^{n+1} h_i] \end{cases}$$

Since the fractal is invariant with respect to the projective transformation

$$R([h_1 : h_2 : \dots : h_n : h_{n+1}]) = [2h_{n+1} - \sum_{i=1}^{n+1} h_i : h_2 : \dots : h_n : h_{n+1}]$$

corresponding to the exchange of the vectors e_1 and e_{n+1} , we can replace $\psi_{n+1}^k \circ \psi_1(F)$ with $\psi_{n+1}^k \circ \psi_1 \circ R(F)$.

Then

$$\psi_1(R([h_1 : h_2 : \dots : h_n : h_{n+1}])) = [h_{n+1} : h_2 : \dots : h_n : \sum_{i=1}^{n+1} h_i]$$

and finally

$$\psi_{n+1}^k \left(\psi_1(R([h_1 : h_2 : \dots : h_n : h_{n+1}])) \right) = [h_{n+1} : h_2 : \dots : h_n : \sum_{i=1}^{n+1} h_i + k \sum_{i=2}^{n+1} h_i].$$

In the chart $v_i = h_i/h_{n+1}$ the map $f_k = \psi_{n+1}^k \circ \psi_1 \circ R$ is represented by

$$f_k(v_1, \dots, v_n) = (1/D, v_2/D, \dots, v_n/D), D = 1 + \sum_{i=1}^n v_i + k(1 + \sum_{i=2}^n v_i).$$

A direct computation shows that the Jacobian of f_k is given by

$$\left| \det \left(\frac{\partial f_k^i}{\partial v_j} \right) \right| = \frac{1}{D^{n+1}}$$

so that $\hat{\mu}(F_1^{(k)}) \leq c_{k-1}^{(n)} \hat{\mu}(F)$ for

$$c_k^{(n)} = \max_{(v_i) \in S(\mathcal{E})} \left| \det \left(\frac{\partial f_k^i}{\partial v_j} \right) \right| \frac{(1 + \sum_{i=1}^n v_i)}{(1 + \sum_{i=1}^n f_k^i(v_i))} = \max_{(v_i) \in S(\mathcal{E})} \frac{(1 + \sum_{i=1}^n v_i)^{n+1}}{D \cdot (v_1 + (k+2) \sum_{i=2}^n v_i)^{n+1}}.$$

As shown in Lemma 2

$$c_k^{(n)} = \frac{2^n}{(2+k)(3+k)^n}$$

with the sole exception of the case $n = 2$, $k = 0$, in which case $c_0^{(2)} = 1/4$. If $n = 2$ then, as already shown in [DD09],

$$\sum_{k=0}^{\infty} c_k^{(2)} = 1/4 + \sum_{k=1}^{\infty} \frac{2^2}{(2+k)(3+k)^2} = \frac{253}{36} - \frac{2}{3}\pi^2 \simeq 0.45 < \frac{1}{2}.$$

In the $n > 2$ case instead we use the fact that

$$\begin{aligned} \sum_{k=0}^{\infty} c_k^{(n)} &= \sum_{k=0}^{\infty} \frac{2^n}{(2+k)(3+k)^n} < \frac{2^{n-1}}{3^n} + 2^n \int_0^{\infty} \frac{dx}{(2+x)(3+x)^n} = \\ &= \frac{2^{n-1}}{3^n} + 2^n \left[\int_0^{\infty} \left(\frac{1}{2+x} - \frac{1}{3+x} \right) dx - \sum_{k=2}^n \int_0^{\infty} \frac{dx}{(3+x)^k} \right] = \\ &= \frac{2^{n-1}}{3^n} + 2^n \left[\ln \frac{3}{2} - \sum_{k=1}^{n-1} \frac{1}{k3^k} \right]. \end{aligned}$$

By Taylor's expansion theorem applied to $\log(1-x)$ we know that there exist a $\xi \in (0, 1/3)$ such that

$$\ln \frac{3}{2} = \sum_{k=1}^n \frac{1}{k3^k} + \frac{1}{(n+1)3^{n+1}(1-\xi)^{n+1}} < \sum_{k=1}^n \frac{1}{k3^k} + \frac{1}{(n+1)2^{n+1}}$$

so that finally

$$\sum_{k=0}^{\infty} c_k^{(n)} < \frac{2^n}{3^n} \left(\frac{1}{2} + \frac{1}{n} \right) + \frac{1}{2(n+1)}.$$

It is easy to verify that the analytical function $g(x) = (\frac{2}{3})^x (\frac{1}{2} + \frac{1}{x}) + \frac{1}{2(x+1)}$ is bigger than $h(x) = 1/x$ for $x \geq 4$, which proves that $\sum_{k=0}^{\infty} c_k^{(n)} < 1/n$ for all $n \geq 4$.

We complete the proof by verifying the case $n = 3$ by a direct computation:

$$\sum_{k=0}^{\infty} c_k^{(3)} = \sum_{k=0}^{\infty} \frac{2^3}{(2+k)(3+k)^3} = 13 - \frac{4}{3}\pi^2 - 8\zeta(3) \simeq 0.22 < \frac{1}{3}.$$

□

Next Corollary will be used later to justify one of the numerical methods we used to evaluate the box-counting dimension of the fractal. An illustration of it can be found in Fig. 4.

Corollary 1. *The fractal set $F(\mathcal{E})$ is contained in the set of accumulation points of the set of barycenters. In particular, the closure of the set of the barycenters is equal to the union of $F(\mathcal{E})$ with the boundaries of the bodies $Z(\mathcal{E}_I)$, $\mathcal{E}_I \in T(\mathcal{E})$.*

Proof. Since $F(\mathcal{E})$ has zero measure it cannot contain any open set. In other words, every open set inside $S(\mathcal{E})$ either is contained inside a body $Z(\mathcal{E}_I)$ for some multi-index I or contains one of them. Let $p \in F(\mathcal{E})$. Then any open neighborhood of p is not contained inside a body and therefore contains one. Inside every body lies a barycenter and so $F(\mathcal{E})$ is contained in the closure of the (countable) set of barycenters. □

In order to study the asymptotics of the fractal it is convenient to pose the following definition:

Definition 2. *We call “section” of an infinite tree T a sequence $\{t_i\}_{i \in \mathbb{N}} \subset T$ such that each element t_n (except for the first) is child of its antecedent t_{n-1} .*

In the Sierpinski case the asymptotics properties do not depend on the particular section but in case of $F(\mathcal{E})$ they do. E.g. consider an edge t of $T(\mathcal{E})$, i.e. a section $t = \{t_k\}$ defined by $t_k = \underbrace{\mathcal{E}_{ii \dots i}}_k$ for some index i . Then the volume

of the simplices $S(t_k)$ decreases polynomially with k , while in the Sierpinski case they always decrease exponentially. Below we study the sections where the volumes grow faster. They are related to n -bonacci sequence, namely sequences whose k -th element is equal to the sum of the previous n ones, thanks to the next proposition:

Proposition 3. *Let A be the set of bodies having non-empty intersection with $Z_I(\mathcal{E})$ and B the set of indices of the hyperfaces of $S(\mathcal{E})$ (labeled after the index of the vertex opposite to it) having points in common with $Z_I(\mathcal{E})$. Then $b(Z_I(\mathcal{E})) = \sum_{z \in A} b(z) + \sum_{k \in B} b_k$, where $b_k = \sum_{i=1}^{n+1} e_i - ne_k$.*

Proof. Since the fractal is invariant under the ψ_i and they are induced by linear transformations, it is enough to prove this property for the barycenter b of the root cut-out polytope $Z = Z(\mathcal{E})$, which cuts all faces of $S = S(\mathcal{E})$. Using the ψ_i^{-1} it is easy to determine that the body corresponding to the k -th face of S is the body of the simplex of vertices

$$\{[e_1 - e_k], \dots, [e_{k-1} - e_k], [e_k], [e_{k+1} - e_k], \dots, [e_{n+1} - e_k]\}$$

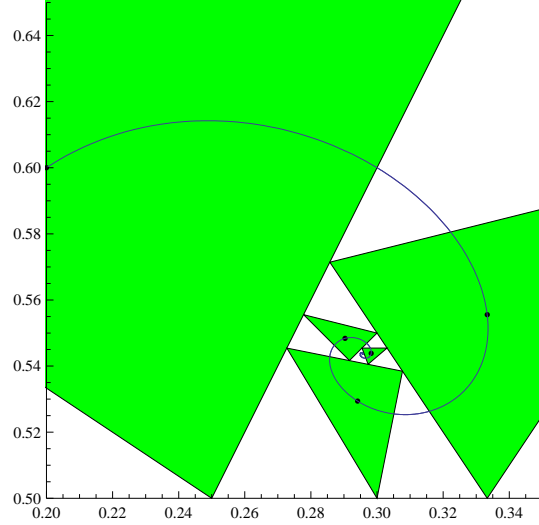


Figure 1: Detail, in the $h_3 = 1$ chart, of the first few bodies corresponding to a Tribonacci section starting from the root of the tree $T(\mathcal{E})$, where $\mathcal{E} = \{(1, 0, 1), (0, 1, 1), (0, 0, 1)\}$. The first barycenter $b_1 = (1, 1, 3)$ of the section is not shown. The next five ones, whose projection on \mathbb{RP}^2 is shown above, are $b_{12} = (1, 3, 5)$, $b_{123} = (3, 5, 9)$, $b_{1231} = (5, 9, 17)$, $b_{12312} = (9, 17, 31)$ and $b_{123123} = (17, 31, 57)$. The centers of the bodies of the section lie on a smooth “Tribonacci projective spiral” drawn above which is winding about $(1/\alpha_3, 1/\alpha_3^2) \simeq (.296, .544)$.

and therefore its barycenter is the vector $b_k = \sum_{i=1}^{n+1} e_i - n e_k$. Now it is easy to verify that

$$\sum_{k=1}^{n+1} b_k = \sum_{k=1}^{n+1} \left(\sum_{i=1}^{n+1} e_i - n e_k \right) = (n+1) \sum_{i=1}^{n+1} e_i - n \sum_{k=1}^{n+1} e_k = \sum_{i=1}^{n+1} e_i = b.$$

□

This result suggests the following interesting way of building sections of a tree $T(\mathcal{E})$ for a basis \mathcal{E} of \mathbb{R}^{n+1} . Pick any element $t_1 = \mathcal{E}_{i_1}$ and continue the section recursively by taking $t_j^\pm = \mathcal{E}_{i_1, i_1 \pm 1, \dots, i_1 \pm j}$, where all indices are meant modulo $(n+1)$. By construction, the body of t_{n+2}^\pm touches the bodies of all of the previous elements of t_i^\pm and therefore its barycenter is given exactly by the sum of the barycenters of their bodies, and the same happens for all remaining terms t_i^\pm , $i > n+2$. We call *Fibonacci sections* this particular kind of sections because the sequence of the corresponding barycenters is a n -bonacci sequence. Fibonacci sections of $T(\mathcal{E})$ are relevant for two reasons: 1. they represent the sections with faster growth high barycenters norms grow faster; 2. they provide a way to get explicit expressions for points in $F(\mathcal{E})$.

Theorem 2. *Barycenters of t_k 's bodies in a Fibonacci sequence grow in norm as α^k , where α is the $(n+1)$ -bonacci number (i.e. the highest module root of the equation $x^{n+1} = x^n + \dots + x + 1$). This is the highest growth rate for barycenters' norms on a section of $T(\mathcal{E})$.*

Proof. We can assume without loss of generality that e_i is the canonical basis for \mathbb{R}^{n+1} , since asymptotics will not change under the action of a single invertible linear transformation, and we can prove the result using the norm $\|v\|_1 = \sum_{i=1}^{n+1} |v_i|$ because in finite dimension all norms are equivalent.

It is well known that the k -th term, $k > n+1$, of a $(n+1)$ -bonacci sequence can be expressed as a linear combination with constant coefficients of the k -th powers of the $n+1$ complex roots of the $(n+1)$ -bonacci equation $x^n = x^{n-1} + \dots + x + 1$. The highest module root is known to be real and it is called $(n+1)$ -bonacci constant. Asymptotically only the highest module root is relevant and this proves the first part of the theorem.

Now, assume that up to the n -th recursive step it happens that at each step k the bodies with higher baricentric norm are the ones built starting from $Z(\mathcal{E})$ and belonging to a Fibonacci section: then at the following recursive step the bodies with higher barycentric norm are exactly the ones which continue those Fibonacci sections. Indeed no body can touch more than one body from each tree level since bodies corresponding to the same level belong to distinct simplices; hence at the $(k+1)$ -th level the bodies' barycenters of the members of those Fibonacci sections are obtained by summing of the highest norm barycenters and the components are all positive, so their norm is the biggest achievable. \square

Remark 2. *Proposition 1, applied to Fibonacci sections, grants that the limit point of a Fibonacci section must belong to $F(\mathcal{E})$. Consider for example the Fibonacci sequence generated by*

$$b_{-n} = (1, \dots, 1, 1-n), \dots, b_0 = (1-n, 1, \dots, 1), b_1 = (1, \dots, 1)$$

In this case all components follow the very same sequence but the component j is shifted by one with respect to the component $j+1$ for $j = 1, \dots, n$, namely $b_k^j = b_{k-1}^{j+1}$. The last component b_k^{n+1} has "initial conditions" $b_{-n}^{n+1} = -n$, $b_{-n+1}^{n+1} = 1$, \dots , $b_0^{n+1} = 1$, so that the first terms of the sequence are $b_1^{n+1} = 1$, $b_2^{n+1} = n+1$, $b_3^{n+1} = 2n+1$ and so on. Since the k -th term of a n -bonacci sequence behaves asymptotically like α^k , in \mathbb{RP}^n the sequence of the corresponding points converges to $(1 : \alpha : \dots : \alpha^n)$.

The following theorems shows that edges and Fibonacci sections are respectively the slower and faster sections with respect to volumes' growth.

Theorem 3. *Let \mathcal{E} be a basis of \mathbb{R}^{n+1} , $T(\mathcal{E}) = \{t_I\}$ its tree of bases and $B(\mathcal{E}) = \{b_I\}$ the corresponding tree of barycenters. Then there exist real constants A, B such that*

$$A|I| \leq \|b_I\| \leq B\alpha^{|I|}$$

for all multiindices I .

Proof. We can prove without loss of generality the theorem by fixing the basis as the canonical basis of \mathbb{R}^{n+1} and the norm as the maximum norm $\|v\|_\infty = \max |v_i|$.

As shown in Theorem 2, the biggest barycenters at every level k are those belonging to a Fibonacci section starting by the root element of the tree; the explicit expression for those sections, modulo permutations, is $b_k = (a_{k-n-1}, \dots, a_k)$, $k > n+1$, where $a_k = \sum_{i=1}^{n+1} \lambda_i \alpha_i^k$, the α_i are the root of the $(n+1)$ -bonacci equation and $\lambda_i = 1/\prod_{j \neq i} (\alpha_i - \alpha_j)$. We order the roots so that $\alpha_1 = \alpha$ is the $(n+1)$ -bonacci constant. Hence, for k big enough,

$$\|b_k\|_\infty = |a_k| \leq 2\lambda_1 \alpha^k$$

The slowest growth, again modulo permutations, is obtained by those n -simplices corresponding to the bases $\{e_1, e_2 + ke_1, \dots, e_{n+1} + ke_1\}$, whose barycenter $b_k = nke_1 + \sum_{i=1}^{n+1} e_i$ has norm $\|b_k\|_\infty = nk + 1$. \square

Now we provide bounds for the volumes of the bodies $Z(\mathcal{E}_I)$ in terms of the norms of the barycenters.

Lemma 3. *Let $W = (w_1, \dots, w_{n+1}) \in \mathbb{R}^{n+1}$, $n > 1$, be a vector with non-negative components and let us build out of it a tree $T(W)$ using the same algorithm used to build $T(\mathcal{E})$, so that e.g. at the first tree level we find $W_1 = (w_1, w_1 + w_2, \dots, w_1 + w_{n+1})$ and the other n vectors obtained similarly. Then if the components of W satisfy the inequalities*

$$\sum_{j \neq j_1, j_2} w_j \leq (n-1)(w_{j_1} + w_{j_2}) \quad (1)$$

the same inequalities hold for all other vectors of the tree.

Proof. We prove the lemma by induction. Let us assume that the inequality is valid for all vectors up to the k -th tree level and be $W' = (w'_1, \dots, w'_{n+1})$ one of the vectors at the level k . For the symmetry of the problem it is enough to verify that the inequality remains true for its first child $W'' = W'_1$ and it is enough to check it in any two cases when its first component w''_1 appears on the right side of the inequality and when it does not.

In the first case let us assume $j_1 = 1$ and $j_2 = 2$. Then the inequality reads

$$\sum_{j=3}^{n+1} w''_j \leq (n-1)(w''_1 + w''_2)$$

that is equivalent to $(n-1)w'_1 + \sum_{j=3}^{n+1} w'_j \leq (n-1)(2w'_1 + w'_2)$ and therefore to $\sum_{j=3}^{n+1} w'_j \leq (n-1)(w'_1 + w'_2)$ which holds by the inductive hypothesis.

In the second case let us assume $j_1 = 2$ and $j_2 = 3$. Then the inequality reads

$$w''_1 + \sum_{j=4}^{n+1} w''_j \leq (n-1)(w''_2 + w''_3)$$

that is equivalent to $(n-1)w'_1 + \sum_{j=4}^{n+1} w'_j \leq (n-1)(2w'_1 + w'_2 + w'_3)$ and therefore to $w'_1 + \sum_{j=4}^{n+1} w'_j \leq (n-1)(w'_1 + w'_2 + w'_3) + w'_1$ which is true because, by the induction hypothesis,

$$w'_1 + \sum_{j=4}^{n+1} w'_j \leq (n-1)(w'_2 + w'_3) \leq (n-1)(w'_1 + w'_2 + w'_3) + w'_1.$$

All remaining inequalities are obtained by permuting the indices. \square

Theorem 4. *For every basis \mathcal{E} of \mathbb{R}^{n+1} there exist real constants A, B such that*

$$\frac{A}{\|b_I\|^{n+1}} \leq \mu(Z_I) \leq \frac{B}{\|b_I\|^{n+1}}$$

for almost all multi-indices I .

Proof. For this proof's sake it is convenient to use the same base \mathcal{E} and measure μ of Theorem 1 and the maximum norm for the barycenters.

These choices have some important advantages: 1. if we call e_i^j , $j = 1, \dots, n+1$, the components of the vectors $\{e_i\} = \mathcal{E}$ with respect to the canonical basis of \mathbb{R}^{n+1} , then the vector $W = (e_1^{n+1}, \dots, e_{n+1}^{n+1})$ built with the $(n+1)$ -th coordinates of the basis vectors changes, when passing from the basis \mathcal{E}_I to $\mathcal{E}_{I, i_{k+1}}$, with the same rule illustrated in the Lemma above and satisfies the set of inequalities (1); 2. if $[h_1 : \dots : h_{n+1}]$ are the canonical homogeneous coordinates for \mathbb{RP}^n , $S(\mathcal{E})$ is entirely contained in the open set $h_{n+1} \neq 0$; 3. the component e_i^{n+1} is not smaller than any other component for every $i = 1, \dots, n+1$; 4. the expressions for the volume of $S(\mathcal{E})$ and $Z(\mathcal{E})$ are particularly simple.

Now let $(x_i^1, \dots, x_i^{n+1})$ be the components of the vectors of the basis \mathcal{E}_I , so that the homogeneous coordinates of the $(n+1)$ vertices of the n -simplex S_I will be $A_i = [x_i^1 : \dots : x_i^{n+1}]$ and those of its body Z_I will be $B_{ij} = [x_i^1 + x_j^1 : \dots : x_i^{n+1} + x_j^{n+1}]$. A direct computation shows that

$$\hat{\mu}(S_I) = \frac{1}{n! \prod_{i=1}^{n+1} x_i^{n+1}}$$

and

$$\hat{\mu}(Z_I) = \frac{1}{n!} \sum_{S \in \mathfrak{S}_n} \frac{1}{\prod_{x_i \neq x_j \in S} (x_i^{n+1} + x_j^{n+1})}$$

where \mathfrak{S}_n is the subdivision of Z_I in $n+1$ simplices $S_I^{(k)}$, where each $S_I^{(k)}$ has the same vertices of S_i except for the k -th vertex, which is replaced by the barycenter of S_I .

Let us consider now one of the simplices $s \in \mathfrak{S}_n$ and let $[e_{i,j,1} + e_{i,j,2}]$, $j = 1, \dots, n+1$, be its vertices. Note that, since all components of the basis vectors

are positive and no component is bigger than the last one, the barycenter's norm is

$$\|b_I\|_\infty = \sum_{l=1}^{n+1} x_l^{n+1}.$$

Hence

$$1 \leq \frac{\|b_I\|_\infty^{n+1}}{\prod_{j=1}^{n+1} (x_{i_{j,1}}^{n+1} + x_{i_{j,2}}^{n+1})} \leq \prod_{j=1}^{n+1} \left(1 + \frac{\sum_{j \neq j_1, j_2} x_j^{n+1}}{x_{i_{j,1}}^{n+1} + x_{i_{j,2}}^{n+1}} \right) \leq n^{n+1}.$$

Since we never used in our calculation the particular choice of the indices for the simplex s , these bounds are valid for all of them and therefore

$$\frac{n+1}{n! \|b_I\|_\infty^{n+1}} \leq \hat{\mu}(Z_I) \leq \frac{(n+1)n^{n+1}}{n! \|b_I\|_\infty^{n+1}}.$$

□

Note that the inequality above does not hold for the n -simplices S_I : for example, in the basis \mathcal{E} used above the simplices corresponding to the bases

$$\mathcal{E}_k = \{e_1 + ke_{n+1}, \dots, e_n + ke_{n+1}, e_{n+1}\}$$

have barycenter $b_k = (1, \dots, 1, nk+1)$ and volume

$$\mu_k = \frac{1}{n! x_1^{n+1} \dots x_{n+1}^{n+1}} = \frac{1}{n! (k+1)^n}$$

which therefore is asymptotic to $1/\|b_k\|^n$ rather than to $1/\|b_k\|^{n+1}$.

Numerical and analytical facts suggest that bodies' diameters are bound by the inverse of their barycenters' norm; in particular it is known to be true for $n = 2$ thanks to an indirect proof (see Section 4.2) and it is confirmed by numerical exploration of the $n = 3$ case (see Section 4.3). We are led therefore to the following conjecture:

Conjecture 1. *For every basis \mathcal{E} of \mathbb{R}^{n+1} there exist constants A and B such that*

$$\frac{A}{\|b(Z)\|^{\frac{n+1}{n}}} \leq |Z| \leq \frac{B}{\|b(Z)\|}$$

where Z is any body associated to the tree $T(\mathcal{E})$ and $|Z|$ its diameter with respect to the canonical distance $d([x], [y]) = \arccos \frac{\langle x, y \rangle}{\|x\| \|y\|}$.

As for the fractal dimension of $F(\mathcal{E})$, we could not find any way to evaluate exact non-trivial bounds for it; in next section we present the numerical evaluation of it for the cases $n = 2, 3$.

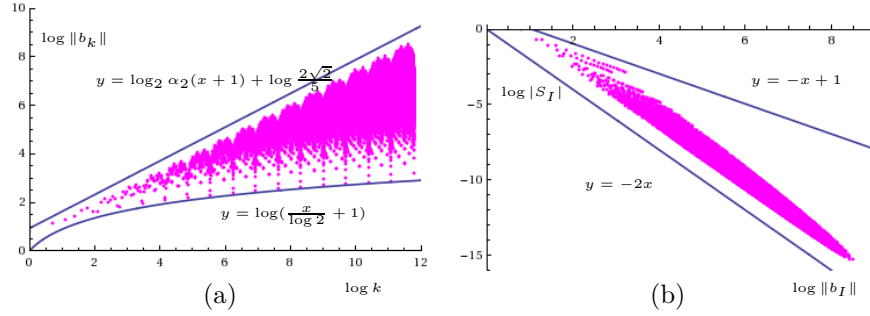


Figure 2: (a) log-log plot of the barycenters' norm vs. its index. The b_I are arranged in the sequence naturally associated to the ordered tree $T(\mathcal{E})$, namely b_I follows b_J if $|I| > |J|$ or, in case the multi-indices have the same order, the lowest index which is different between I and J is bigger in I . Since there are 2^k nodes at the level k the upper and lower bounds are evaluated using the fact that $k+1 \leq \|b_{2^k-1}\| \leq \frac{2\sqrt{2}}{\sqrt{5}}\alpha_2^{k+1}$ and therefore $\log_2 k \leq \|b_k\| \leq \frac{2\sqrt{2}}{\sqrt{5}}(k+1)^{\log_2 \alpha_2}$. (b) log-log plot of the length of the 1-simplices S_I vs. the barycenters' norm.

4 Analysis of the cases $n = 1, 2, 3$

4.1 The case $n=1$

The construction we discussed above does not strictly speaking apply to the $n = 1$ case. E.g. bodies here are simply single points, Theorem 3 does not apply and all asymptotics about the measures of bodies have no meaning here. Nevertheless a few things survive: the tree $T(\mathcal{E})$ and its Fibonacci sections can still be built and we can study the asymptotics of the lengths of the 1-simplices.

To begin, let us choose

$$\mathcal{E} = \{e_1 = (1, 0), e_2 = (1, 1)\}$$

The set $F(\mathcal{E})$ is invariant with respect to the projective transformations

$$\psi_1([h_1 : h_2]) = [h_1 + h_2 : h_1], \psi_2([h_1 : h_2]) = [2h_1 - h_2 : h_1]$$

and it is obtained from the segment $[0, 1]$ (in the projective chart $x = 1$) by removing a countable set of infinite (rational) points, so that it has full measure and therefore $\dim_H F(\mathcal{E}) = 1$.

The growth rate of the Fibonacci sections here is given by the Golden Ratio $\alpha = (1 + \sqrt{5})/2$ and for the norm of the sections' barycenters we have the inequalities $k + 2 \leq \|b_k\|_\infty$ for the slowest section and $\|b_k\|_\infty \leq \frac{2}{\sqrt{5}}\alpha^{k+1}$ for the fastest (see fig. 2(a)). In particular the components of the two root Fibonacci sections are exactly the Fibonacci numbers: e.g. taking $b_1 = b(Z_{\mathcal{E}}) = (1, 2)$ and $b_2 = b(Z_{\{e_1, e_1+e_2\}}) = (2, 3)$ we have that $b_3 = (3, 5)$, $b_4 = (5, 8)$ and so on.

Asymptotics of bodies have no meaning here but still we can say something about the asymptotics of the lengths of the 1-simplices constituting the binary

tree $T(\mathcal{E})$. Indeed if $\mathcal{E}' = \{ae_1 + be_2, ce_1 + de_2\}$ with $e_i = (x_i, y_i)$ then, in the chart $y = 1$,

$$\begin{aligned} \mu(\mathcal{E}') &= d([ae_1 + be_2], [ce_1 + de_2]) = \left| \frac{ax_1 + bx_2}{ay_1 + by_2} - \frac{cx_1 + dx_2}{cy_1 + dy_2} \right| \\ &= \frac{|ad - bc| \cdot |x_1y_2 - x_2y_1|}{(ay_1 + by_2)(cy_1 + dy_2)} = \frac{1}{(ay_1 + by_2)(cy_1 + dy_2)} = \frac{1}{(a+b)(c+d)} \end{aligned}$$

where $|x_1y_2 - x_2y_1| = 1$ is the surface of the parallelogram corresponding to \mathcal{E} and $|ad - bc| = 1$ because of the way the algorithm produces the new bases. In our concrete case $y_i = 1$ and therefore

$$\frac{1}{\|b\|_\infty^2} \leq \mu(\mathcal{E}') = \frac{1}{\|b\|_\infty} \left(\frac{1}{a+b} + \frac{1}{c+d} \right) \leq \frac{2}{\|b\|_\infty}$$

Numerical illustrations of this pair of inequalities are shown in fig 2(b).

4.2 The case n=2

This is the only case where the polytopes corresponding to the bases and to the bodies are of the same kind, namely triangles. The algorithm that produces the fractal reduces here to the following:

Algorithm 1.

1. On the three edges of the triangle Δ with vertices $\{[e_i]\}_{i=1,2,3}$ select the three points $f_1 = [e_2 + e_3]$, $f_2 = [e_3 + e_1]$, $f_3 = [e_1 + e_2]$;
2. subtract from Δ the interior of the triangle Z (the “body” of Δ) with vertices $\{f_1, f_2, f_3\}$;
3. repeat recursively the algorithm on each of the three triangles that are left after the subtraction.

Note that no two bodies have in common more than a point, i.e. they meet transversally, so the set $F(\mathcal{E})$ is never empty and actually it contains uncountably many points; countably many of them can be explicitly evaluated through Fibonacci sections of the ternary tree $T(\mathcal{E})$.

Consider for example the case

$$\mathcal{E} = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}.$$

As shown in Theorem 3 the barycenter of every body triangle is the vector sum of the barycenters of the three body triangles it touches with its vertices (note that by construction no two bodies have a vertex in common) and when a body touches one of the sides of the root triangle $S(\mathcal{E})$ then we sum instead the vectors $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$ in correspondence respectively with the sides opposed to the vertices $[e_1]$, $[e_2]$ and $[e_3]$. The barycenters of one of the six root Fibonacci sections are determined by the first elements

$$b_{-3} = (1, 1, -1), b_{-2} = (1, -1, 1), b_{-1} = (-1, 1, 1)$$

so that the generic element of the section is given by $b_k = (a_{k-2}, a_{k-1}, a_k)$, where a_k is the sequence of Tribonacci numbers with initial conditions $a_{-2} = 1$, $a_{-1} = 1$, $a_0 = 1$. The expression of the generic term is given by

$$a_k = \frac{(1-\beta)(1-\bar{\beta})}{(\alpha-\beta)(\alpha-\bar{\beta})}\alpha^k + \frac{(1-\alpha)(1-\bar{\beta})}{(\beta-\alpha)(\beta-\bar{\beta})}\beta^k + \frac{(1-\alpha)(1-\beta)}{(\bar{\beta}-\beta)(\bar{\beta}-\alpha)}\bar{\beta}^k$$

where α , β and $\bar{\beta}$ are the roots of the Tribonacci equation $x^3 = x^2 + x + 1$. Since $|\beta| < \alpha$ we have that $\|b_k\|_\infty \leq \frac{3(\alpha^2-1)}{3\alpha^2-2\alpha-1}\alpha^k$ and the limit point (see fig. 3) is $(1 : \alpha : \alpha^2)$. Note that all barycenters of this sequence lie on the “projective Tribonacci spiral”

$$\gamma(t) = [a(t-2) : a(t-1) : a(t)]$$

where $a(t)$ is the trivial analytical extension of the a_k sequence. The fractal is invariant with respect to the projective transformations

$$\begin{aligned}\psi_1([h_1 : h_2 : h_3]) &= [h_1 + h_2 + h_3 : h_2 : h_3] \\ \psi_2([h_1 : h_2 : h_3]) &= [h_1 : h_1 + h_2 + h_3 : h_3] \\ \psi_3([h_1 : h_2 : h_3]) &= [h_1 : h_2 : h_1 + h_2 + h_3]\end{aligned}$$

so by applying any finite composition of them we obtain countably many explicit points of $F(\mathcal{E})$.

The slowest sections in the barycenters’ norms growth is, modulo indices permutations,

$$t_k = \underbrace{\mathcal{E}_1, \dots, 1}_k = \{e_1^{(k)} = e_1, e_2^{(k)} = e_2 + ke_1, e_3^{(k)} = e_3 + ke_1\}$$

for which $b_k = (2k+3, 1, 1)$ and therefore $\|b_k\|_\infty = 2k+3$.

In figs. 4.2(a-c) we show the numerical results for the asymptotic behaviour of the barycentric norms and the bodies’ surfaces and diameters.

Note that in this particular case Conjecture 1 is known to be true through an indirect proof. Indeed, this fractal comes up naturally in the study of the asymptotics of plane sections of periodic surfaces, which in turn comes from the problem of the motion of quasi-electrons under a strong magnetic field (see [NM03] for a detailed account), in the particular case of the regular triply-periodic skew polyhedron $\{4, 6|4\}$ [DD09]. In that setting the basis is

$$\mathcal{E}_C = \{e_1 = (1, 0, 1), e_2 = (0, 1, 1), e_3 = (1, 1, 0)\}$$

and the barycenter b of a body Z represents a homological discrete “first integral” of a Poisson dynamical system which dictates the asymptotic directions of the plane sections in the following way: the open sections obtained by cutting the polyhedron with planes perpendicular to every direction $\omega \in Z$ are all strongly asymptotic to the direction “ $\omega \times b$ ”. It is a general theorem of that theory the fact that the diameter of a body Z is bounded by $C/\|b(Z)\|$ where C is a constant depending only on the surface [De 05], which then establishes the following theorem for this $n = 2$ case:

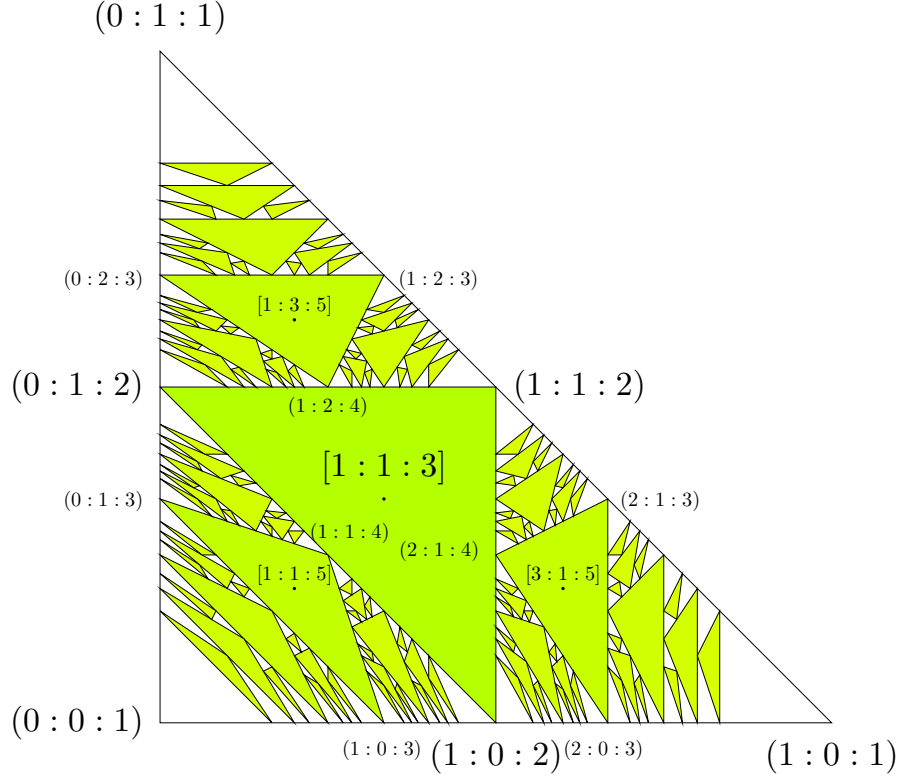


Figure 3: Plot of $F^5(\mathcal{E})$, namely of the bodies up to the forth recursion level, for $\mathcal{E} = \{(1, 0, 1), (0, 1, 1), (0, 0, 1)\}$ in the $h^3 = 1$ projective chart of \mathbb{RP}^2 . Bodies are colored in green, so the points of F^5 are the white ones. The homogeneous coordinates of the vertices of the first and second level bodies are shown together with the body's barycenters, for which we used the square brackets for sake of clarity. Note that barycenters can be obtained in three ways: (i) by summing the barycenters of the three bodies touched by the vertices – note that in case a vertex touches a root simplex edge then the following should be used: $(1, 1, 1)$ for the edge opposite to $[0 : 0 : 1]$, $(1, -1, 1)$ for the one opposite to $[1 : 0 : 1]$ and $(-1, 1, 1)$ for the one opposite to $[0 : 1 : 1]$; (ii) by summing the coordinates of the vertices of the triangle that generated the body; (iii) by summing the coordinates of the vertices of the body and dividing them by 2 – this corresponds to the fact the volume associated to the basis corresponding to the vectors $\{e_1 + e_2, e_2 + e_3, e_3 + e_1\}$ is double with respect to the basis $\{e_1, e_2, e_3\}$.

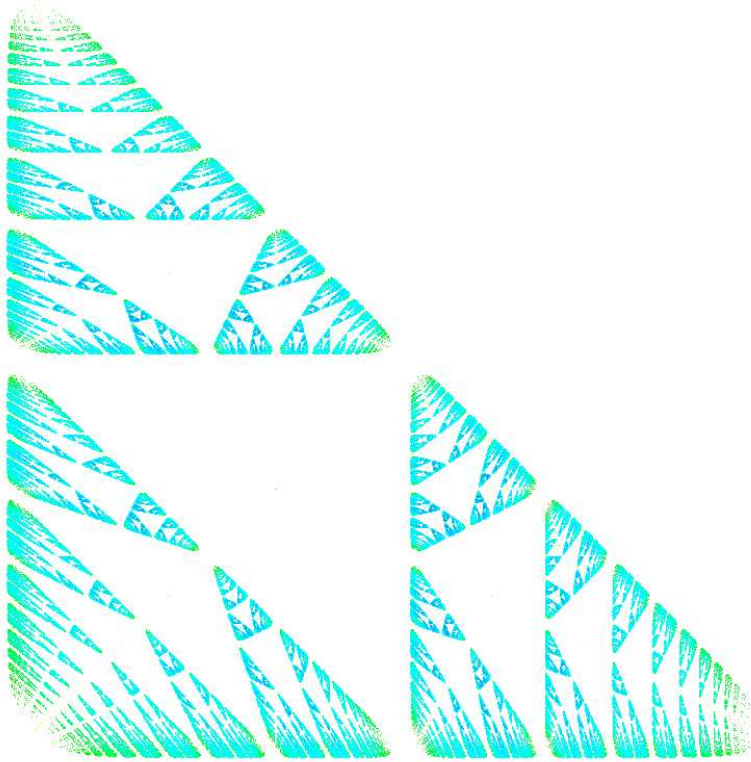


Figure 4: Plot of the barycenters of all 3487590 bodies up to the thirteenth recursion level for $\mathcal{E} = \{(1, 0, 1), (0, 1, 1), (0, 0, 1)\}$ in the $h^3 = 1$ projective chart of \mathbb{RP}^2 . The colors of the points goes from red to blue as the Euclidean norm of the barycenters grows. By Corollary 1 the closure of the set of barycenters is equal to the boundaries of all bodies plus the points of the fractal $F(\mathcal{E})$, so this picture represents an approximation of the real fractal (and actually no point shown belongs to $F(\mathcal{E})$ since barycenters are all contained inside the bodies. Nevertheless they can approximate as close as wished the set $F(\mathcal{E})$ and so they can be used to derive a numerical evaluation of the box-counting dimension of $F(\mathcal{E})$.

Theorem 5. *Let Z be a body in $T(\mathcal{E}_C)$ with area $\mu(Z)$ (where μ is the same measure used in Theorem 4), diameter $|Z|$ and barycenter b . Then the following inequalities hold asymptotically:*

$$\frac{1}{4\sqrt{3}\|b\|^{\frac{3}{2}}} \leq |Z| \leq \frac{6}{\|b\|}, \frac{1}{2\|b\|^3} \leq \mu(Z) \leq \frac{12\sqrt{3}}{\|b\|^3}.$$

Proof. The inequality for the area of Z is just the restriction of Theorem 4 to $n = 2$ together with the fact that $\|b\|_\infty \leq \|b\| \leq \sqrt{3}\|b\|_\infty$. The right hand side for the diameter comes from the general theory of plane sections of a triply periodic surface that, applied to this particular case, states [De 05] that the distance between the barycenter and the bodies' vertices is bounded by $3/\|b\|$, where the 3 is the double of the area of the basic cell of the periodic surface cited above in this section. The left hand side comes simply from the fact that a triangle of area a cannot have a diameter smaller than $\sqrt{2a/\sqrt{3}}$. \square

Being unable to evaluate analytical bounds for the Hausdorff dimension d_C of $F(\mathcal{E}_C)$, we compute numerically four different quantities that may give hints on whether d_C is integer or not (the non-integrality of d_C would confirm a general conjecture by Novikov [NM03]).

First we get a direct upper bound for the Hausdorff dimension by counting the smallest number of squares of side $\epsilon = 2^{-l}$, $l = 0, \dots, 12$, needed to cover F^{12} , i.e. the union of all bodies up to the 12-th order of recursion; as shown in fig. 4.2(h), we get $d_C \lesssim 1.7$.

Then we evaluate the *Minkowsky dimension*, namely the limit

$$2 - \lim_{\epsilon \rightarrow 0} \frac{\log V(F_\epsilon)}{\log \epsilon}$$

where $V(F_\epsilon)$ is the surface of the ϵ neighborhood of F , using the formula [Fal97, Gai06]

$$V(F_\epsilon) = p\epsilon + \epsilon \sum_{i=1}^{k_\epsilon} p_i + A - \sum_{i=1}^{k_\epsilon} a_i + \epsilon^2 \left(\pi - \sum_{i=1}^{k_\epsilon} \frac{p_i^2}{4a_i} \right)$$

where k_ϵ is the integer such that $\rho_{k_\epsilon+1} \leq \epsilon \leq \rho_{k_\epsilon}$, ρ_k is the radius of the inscribed circle to the body Z_k and the bodies are sorted in descending order with respect to the radii. In fig. 4.2(g) we show the numerical results we got by evaluating the volume of the neighborhoods of \mathcal{E} of radii $r_n = 1.2^{-n}$ for $n = 1, \dots, 50$, which suggests a Minkowsky dimension between 1.7 and 1.8.

Next, we evaluate numerically the growth rate of the radii after sorting them in decreasing order (fig. 4.2(d)) and then the corresponding bounds for the bodies areas (fig. 4.2(e)) and diameters (fig. 4.2(f)). In this case we obtain that $\epsilon \asymp k^{-.69}$, $Ak^{-1.45} \leq a_k \leq Bk^{-1.1}$ and $A'k^{-.75} \leq p_k \leq B'k^{-.3}$, so that $A''\epsilon^{.65} \leq V_\epsilon \leq B''\epsilon^{.145}$. From this we get a second evaluation, compatible but much looser, for the Minkowsky dimension: $1.35 \leq \dim_M F(\mathcal{E}_C) \leq 1.86$.

Finally, we use Corollary 1 and evaluate the box-counting dimension of the set $B = \{[b_I(\mathcal{E}_C)]\} \subset \mathbb{RP}^2$, namely the set of barycenters of the bases in the

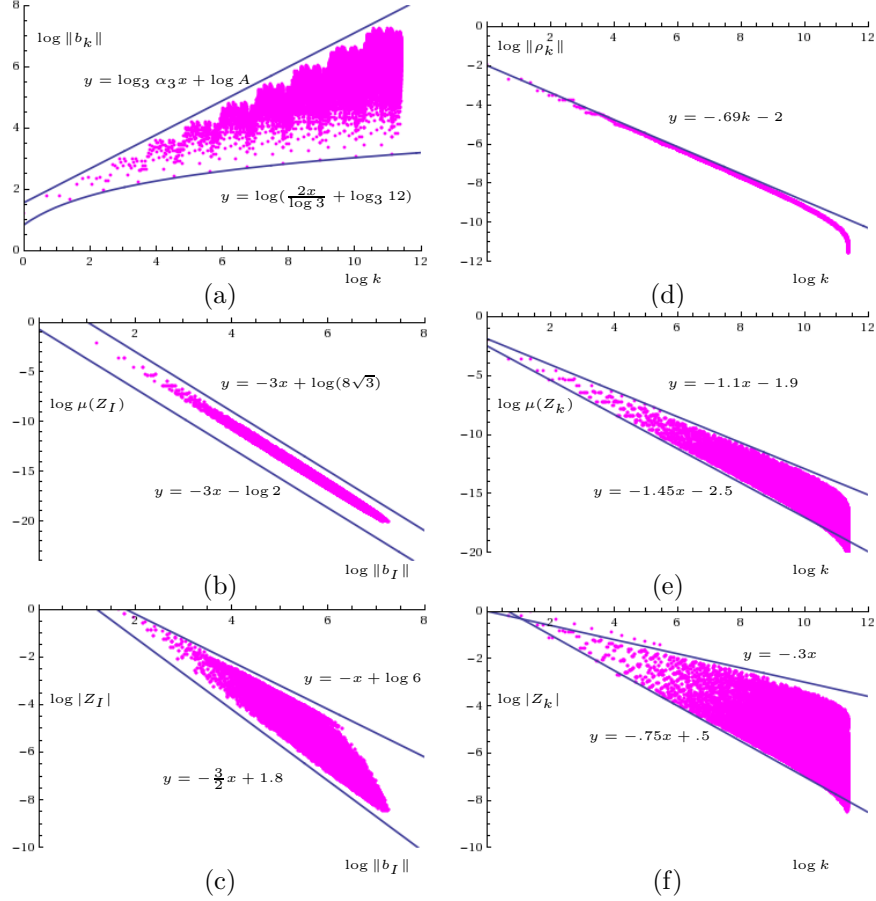


Figure 5: Log-log plots for the main quantities in the $n = 2$ case for $\mathcal{E} = \{(1, 0, 1), (0, 1, 1), (0, 0, 1)\}$ (a) Barycenters norms vs indices – as explained in fig. 2 the b_k are ordered according to the natural order induced by the tree, so that $2k + 3 \leq \|b_{\frac{3k-1}{2}}\| \leq \sqrt{3}\alpha_3^{\frac{3(1-\beta_3)(1-\bar{\beta}_3)}{(\alpha-\beta_3)(\alpha-\bar{\beta}_3)}}\alpha_3^k = A\alpha_3^k$ and therefore $\frac{2}{\log 3} \log k + \log_3 12 \leq \|b_k\| \leq Ak^{\log_3 \alpha_3}$. (b) Bodies' volumes vs barycenters norms and (c) bodies' diameters vs barycenters norms – the lines bounding the numerical data come immediately from the inequalities in Theorem 5. For the next three plots no exact formulae are known so the lines shown represent just an interpolation of the numerical data. (d) Radii of the circles inscribed in the bodies vs k after sorting the radii in descending order. (e) Areas of the bodies and (f) their diameter sorted according with their radii.

tree $T(\mathcal{E}_C)$ considered as points in the projective plane. Since the closure of B is the union of $F(\mathcal{E}_C)$ with the (one-dimensional) boundaries of the bodies, a dimension higher than one must be due to the points in $F(\mathcal{E}_C)$. As shown in Fig. 4.3, the dimension appear to be about 1.69. Notice that applying this method to the Sierpinski triangle, whose dimension is $d_S = \log 3 / \log 2$, gives the correct approximation to the third digit $d_S \simeq 1.59$.

In conclusion, the four evaluations are in excellent agreement with each other and indicate a non-integer Hausdorff dimension for this fractal, probably about 1.7; finding exact bounds would be nicer though since this would represent the first analytical confirmation of a conjecture of Novikov about the non-integer dimension of fractals coming from the theory of asymptotics of plane sections of triply-periodic surfaces.

4.3 The case $n=3$

When $n = 3$, every body has 6 vertices: one for each edge of the tetrahedron they belong to, and eight triangular faces, one for each face and one for each corner of the tetrahedron. Bodies that touch each other share a whole triangle (rather than a single point as in the $n = 2$ case) in the following way: bodies can meet only on the faces that do not come from the tetrahedra S_I and, on those faces, these shared triangles form a fractal of the $n = 2$ kind (see Fig. 4.3).

In the particular case of $\mathcal{E}_T = \{(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$ the barycenter of the root tetrahedron is $(1, 1, 1, 4)$ and the volume inequalities translate in

$$\frac{2}{3\|b_I\|_\infty^4} \leq \mu(Z_I(\mathcal{E}_T)) \leq \frac{2 \cdot 3^3}{\|b_I\|_\infty^4}.$$

In case of barycenters' norms we have $4 + 3k \leq \|b_k\|$ for the slowest tree section and $\|b_k\| \leq 2 \frac{(1-\beta)(1-\tilde{\beta})(1-\gamma)}{(\alpha_4-\beta)(\alpha_4-\tilde{\beta})(\alpha_4-\gamma)} \alpha_4^{k+4}$ for the fastest.

Numerical evaluations of the Hausdorff dimension are more cumbersome for $n = 3$ because the number of bodies grows very large after few iterations of the generating algorithm (getting rather heavy on both CPU and RAM consumption) and their geometry gets much more complicated.

First we evaluate the Minkowsky dimension as the growth rate of the volume V and surface S of the bodies when sorted by the radius $\rho = V/S$. In this case we use the fact that [Gai06]

$$\sum_{i=k_\epsilon+1}^{\infty} V_i \leq V_\epsilon \leq S_\epsilon + H\epsilon^2 + \epsilon \sum_{i=1}^{k_\epsilon} S_i + \sum_{i=k_\epsilon+1}^{\infty} V_i + \frac{4}{3}\epsilon^3$$

where V_ϵ is the volume of the neighborhood of $F(\mathcal{E}_T)$ of radius ϵ , k_ϵ the integer such that $\rho_{k_\epsilon+1} \leq \epsilon \leq \rho_{k_\epsilon}$, V_i and S_i the volume and surface of the body Z_i , S and H the surface and mean curvature of the starting tetrahedron. From the numerical data (see fig. 4.3(a-c)) we obtain that $\epsilon \asymp k^{-.57}$, $Ak^{-.77} \leq V_k \leq Bk^{-1.2}$ and $A'k^{-1.3} \leq S_k \leq B'k^{-.5}$, so that $A''\epsilon^{1.34} \leq V_\epsilon \leq B''\epsilon^{1.25}$ and therefore $1.66 \leq \dim_M F \leq 2.75$. Unfortunately, unlike in the $n = 2$ case, we

are not able to exclude from this bounds that the fractal has integer dimension $\dim_M F = 2$; note that this is exactly what happens for the Tetrix, i.e. the three-dimensional analog of the Sierpinski triangle.

Next we use Corollary 1 and evaluate numerically the box-counting dimension d_{bc} of the set $B = \{[b_I(\mathcal{E}_T)]\} \subset \mathbb{RP}^3$, namely the set of barycenters of the tree $T(\mathcal{E}_T)$ considered as points in the projective three-space. Analogously to the case $n = 2$, the closure of B is the union of $F(\mathcal{E}_T)$ with the (two-dimensional) boundaries of the bodies, a dimension higher than two must be due to the points in $F(\mathcal{E}_T)$. We obtain $d_{bc} \simeq 2.20$ (see Fig. 4.3). Notice that the very same method, applied to the Tetrix, whose dimension is $d_T = 2$, gives the quite close result $d_T \simeq 2.01$.

We could not get useful information from the other two methods used in the $n = 2$ case. In conclusion, the two numerical results we obtained are compatible with each other and the estimate of the box-counting dimension of barycenters is sufficiently far from integer to make us think that, unlike the Tetrix, this fractal may have Hausdorff dimension higher than 2.

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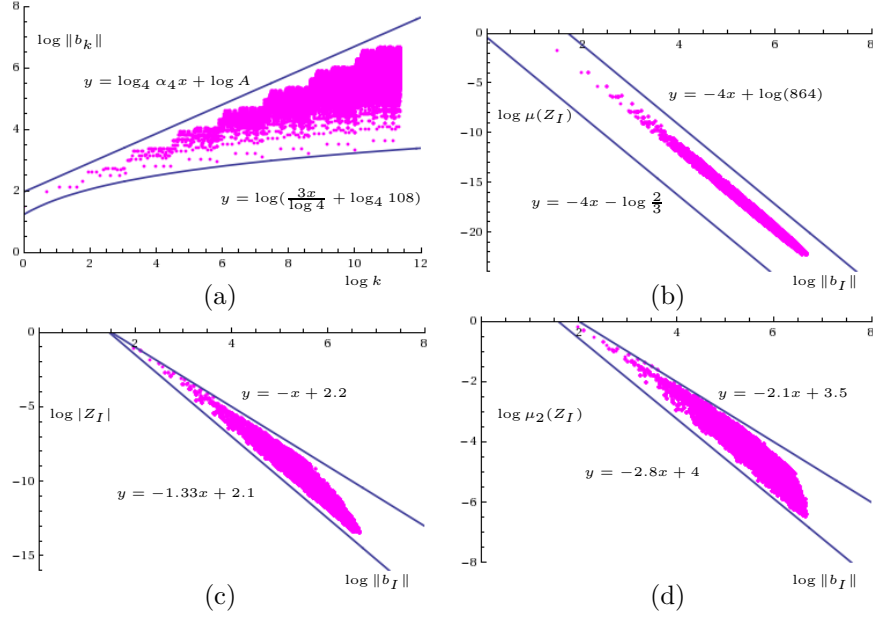


Figure 6: log-log plots for the main quantities in the $n = 3$ case for $\mathcal{E}_T = \{(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$ (a) Barycenters norms vs indices – as explained in fig. 2 the b_k are ordered according to the natural order induced by the tree, so that $3k + 4 \leq \|b_{\frac{4k-1}{3}}\| \leq 2\alpha_3^4 \frac{(1-\beta_4)(1-\beta_4)(1-\gamma_4)}{(\alpha-\beta_4)(\alpha-\beta_3)(\alpha-\gamma_4)} \alpha_4^k = A\alpha_4^k$ and therefore $\frac{3}{\log 4} \log k + \log_4 108 \leq \|b_k\| \leq Ak^{\log_4 \alpha_4}$. (b) Bodies' volumes vs barycenters norms – the lines bounding the numerical data come immediately from the inequalities in Theorem 5. (c) Bodies' diameters vs barycenters norms and (d) bodies' surfaces areas; for these quantities no exact formulae are known so the lines shown represent just an interpolation of the numerical data.

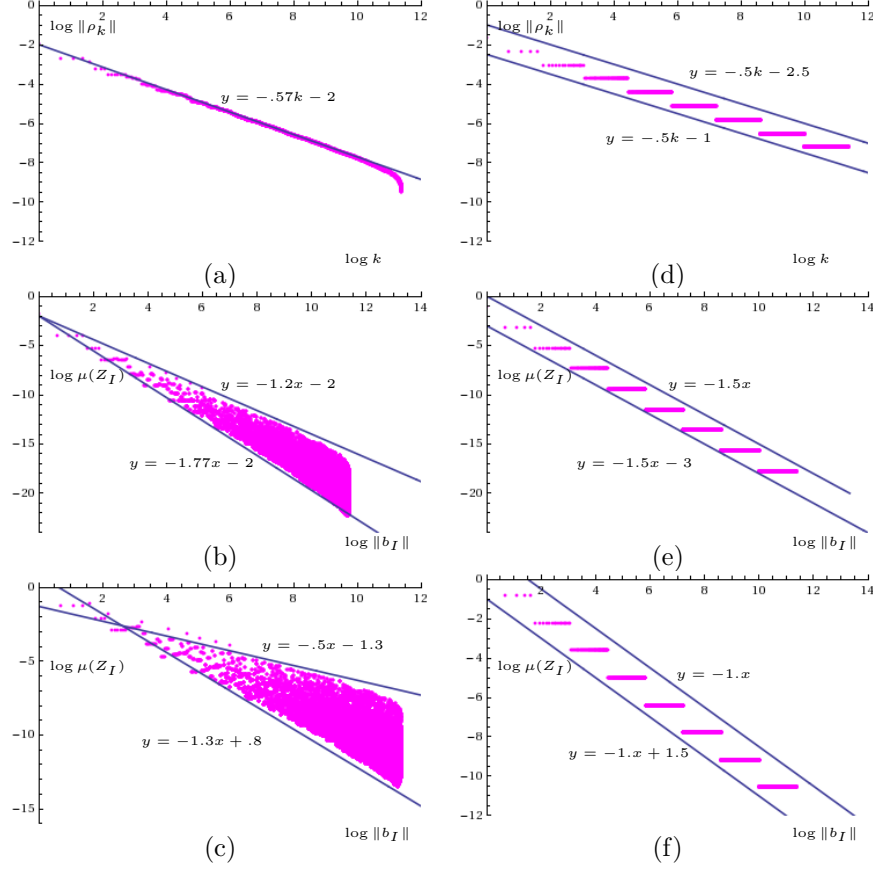


Figure 7: Comparison of the numerical data between the $n = 3$ case $F(\mathcal{E}_T)$ for $\mathcal{E}_T = \{(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$ and the Sierpiński tetrahedron \mathcal{S} about the asymptotic behaviour of the bodies' volumes V and surfaces S sorted by their "radii" $\rho = V/S$ in descending order. No exact formulae are known for these plots so the lines shown above represent just an interpolation of the numerical data. (a,c) Radii of the circles inscribed in the bodies vs k after sorting the radii in descending order for the $n = 3$ case (left) and the Sierpiński tetrahedron (right). (b,d) Areas of the bodies and (c,f) their surface sorted according with their radii for the $n = 3$ case (left) and the Sierpiński tetrahedron (right). From these interpolation we obtain that for $F(\mathcal{E}_T)$ we have $1.6 \leq \dim_B F(\mathcal{E}_T) \leq 2.9$ and for the Sierpiński tetrahedron we get the correct answer $\dim \mathcal{S} = 2$.

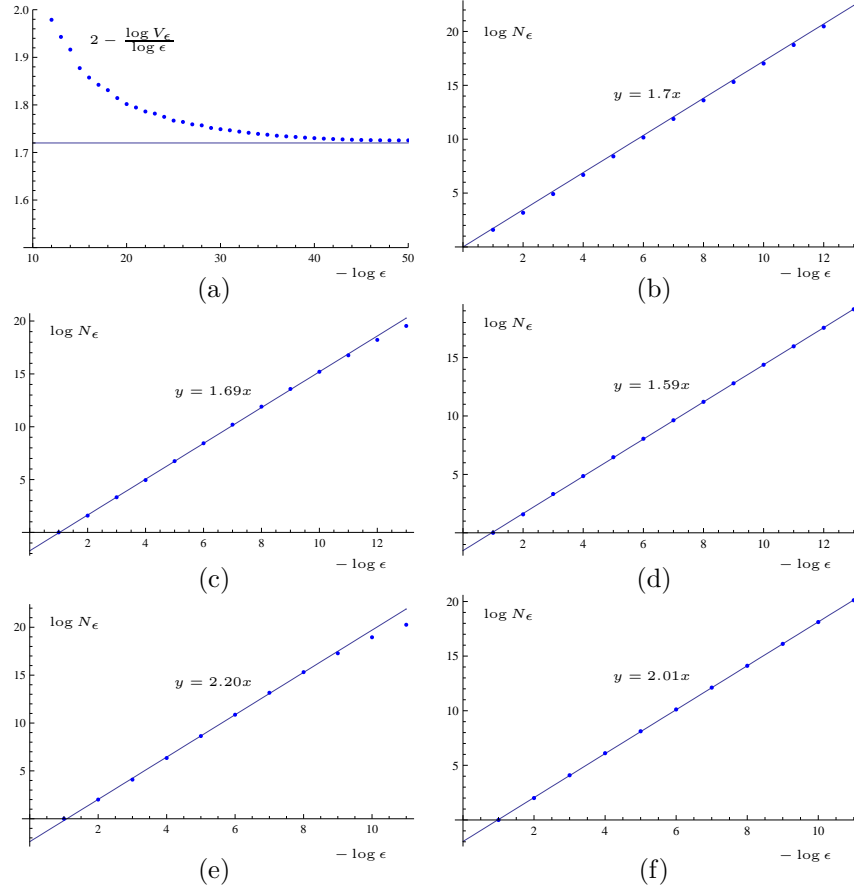


Figure 8: Plots relative to the evaluation of fractal dimensions of $F_C = F(\mathcal{E}_C) \subset \mathbb{RP}^2$ (see Section 4.2) and $F_T = F(\mathcal{E}_T) \subset \mathbb{RP}^3$ (see Section 4.3). (a) Evaluation of the Minkowsky dimension by direct numerical computation of the area V_ϵ of the of the ϵ neighborhood of F_C , based on the fact that, for “nice” fractals F , $\dim F = \lim_{\epsilon \rightarrow 0^+} \left[2 - \frac{\log V_\epsilon}{\log \epsilon} \right]$. (b) Evaluation of the box-counting dimension $d_{bc}(F_C)$ by direct computation of the number of squares N_ϵ needed to cover F^{12} , the 12-th order approximation of F_C , for $\epsilon = 2^{-k}$, $k = 1, \dots, 12$. The data strongly suggest that $d_{bc}(F_C) \simeq 1.7$. (c) Evaluation of the box-counting dimension $d'_{bc}(F_C)$ of the set of barycenters (considered as points in \mathbb{RP}^2) of the bodies $Z(\mathcal{E}_I)$, $\mathcal{E}_I \in T(\mathcal{E}_C)$. By Corollary 1, the closure of this set is the union of $F(\mathcal{E}_C)$ with a 1-dim. set. We get, in excellent agreement with (b), $d'_{bc}(F_C) \simeq 1.69$. (d) Same evaluation as in (c) in case of the Sierpinski triangle. The estimated dimension $d'_{bc} \simeq 1.59$ is in perfect agreement with the exact result $\log 3 / \log 2 \simeq 1.585$. (e) Evaluation of the box-counting dimension $d'_{bc}(F_T)$ of the set of barycenters (considered as points in \mathbb{RP}^3) of the bodies $Z(\mathcal{E}_I)$, $\mathcal{E}_I \in T(\mathcal{E}_T)$. By Corollary 1, the closure of this set is the union of $F(\mathcal{E}_T)$ with a 2-dim. set. We get $d'_{bc}(F_T) \simeq 2.20$. (f) Same evaluation as in (e) in case of the Tetrix (three-dim. analogue of the Sierpinski triangle). The estimated dimension $d'_{bc} \simeq 2.01$ is in excellent agreement with the exact result 2.

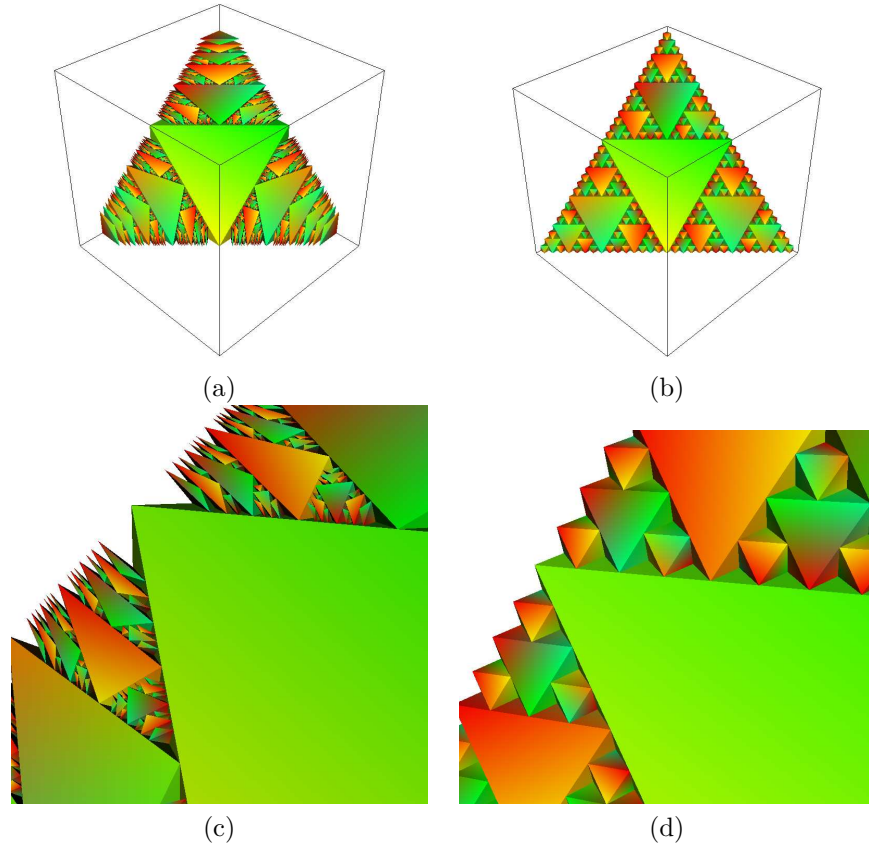


Figure 9: (a,c) Total view and detail of $F^6(\mathcal{E})$, namely of the bodies up to the fifth recursion level, for $\mathcal{E} = \{(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$ in the $h^4 = 1$ projective chart of \mathbb{RP}^3 . The bodies of $S(\mathcal{E})$ are shown, up to the fifth recursion level, in red-green colors; the points of $F^6(\mathcal{E})$ are their complement in the tetrahedron of vertices (in the chart $h^4 = 0$) $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. (b,d) Total view and detail, up to the fourth recursion level, of the Sierpiński tetrahedron.

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